

Lectures on Scattering Amplitudes via AdS/CFT

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We review recent progress on computing scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills at strong coupling by using the AdS/CFT duality. We do explicit computations by using both, dimensional regularization and a cut-off in the radial direction. Up to an additive constant independent on the kinematics, the finite piece of the amplitude is the same in both regularizations. The later scheme is particularly appropriate for understanding the conformal properties of the amplitudes.¹

¹ Based on lectures given by the author at several places.

1. Introduction

In these notes we study gluon scattering amplitudes of planar maximally supersymmetric, $\mathcal{N} = 4$, Yang-Mills (MSYM). In this theory one can go much further in the computation of scattering amplitudes than in other gauge theories, such as QCD, but at the same time we expect that these amplitudes can teach us something about QCD amplitudes. On one hand, perturbative computations are simpler. In the last few years there have been an enormous progress in the computation of MSYM scattering amplitudes. On the other hand, the strong coupling limit of the theory can be studied by means of the *AdS/CFT* duality by considering a dual weakly coupled string sigma model.

The main aim of these notes is to report on recent progress in the application of the *AdS/CFT* duality to compute gluon scattering amplitudes of planar MSYM at strong coupling [1] [2] .

These notes are organized as follows. In the next section we briefly describe some weak coupling perturbative results for the scattering amplitudes in planar MSYM. In section three we explain how the *AdS/CFT* duality can be used to compute scattering amplitudes at strong coupling. The amplitudes are IR divergent, so a regulator needs to be introduced in order to define them properly. We start by introducing a D-brane as infrared (IR) regulator in order to set-up our calculation. Actual computations, however, are done in the super-gravity analog of dimensional regularization, since it is easier to proceed in this scheme and besides, we want to compare our results with the perturbative results which were computed using dimensional regularization. We show in some detail how our prescription works for the scattering of four gluons and repeat the same computation by using a cut-off in the radial direction as IR regulator. At the end of section three we emphasize the role of a dual conformal symmetry and present a simple argument leading to a (dual) conformal Ward identity which fixes the form of the amplitude for the scattering of four and five gluons. In section four we review a recent conjectured duality between scattering amplitudes and Wilson loops and we use the proved one loop duality in order to test a particular guess for the form of the scattering amplitudes. Finally we end up with some conclusions and a list of open problems and future directions.

2. Perturbative planar MSYM scattering amplitudes

In this section we briefly discuss the progress during the last few years in computing perturbative planar scattering amplitudes of MSYM.²

Gluon states $|\mathcal{G}\rangle = |h, p^\mu, a\rangle$ are characterized by their helicity $h = \pm 1$, four momenta p^μ and color indices a in the adjoint representation. Generic amplitudes depend in a complicated manner (even at tree level!) on these.

In the limit of a large number of colors it is useful to write the amplitudes into a color decomposed form. Denoting by $\mathcal{A}_n^{(L)}$ the L -loop, n -point amplitude we have

$$\mathcal{A}_n^{(L)} \approx g^{n-2} (g^2 N)^L \sum_{\rho} \text{Tr}(T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \dots, \rho(n)) \quad (2.1)$$

where the sum runs over non cyclic permutations, N denotes the number of colors and g the gauge theory coupling constant. This decomposition clearly separates the color structure from the kinematics. The leading color ordered amplitude $A_n^{(L)}$ will hence depend only on the momenta and the helicities of the particles undergoing the scattering.

On shell amplitudes of massless gauge theories in four dimensions, such as *MSYM*, contain IR singularities.³ Hence, in order to define them, one needs to introduce a regulator. Commonly, a version of dimensional regularization is used, *i. e.* we consider the theory in $D = 4 - 2\epsilon$ dimensions. For instance, the one loop scattering amplitude for four gluons contains a factor of the form

$$I_4^{(1)} = \int \frac{d^D p}{p^2(p - k_1)^2(p - k_1 - k_2)^2(p + k_4)^2} \quad (2.2)$$

We recognize two kind of divergences. First, from the region $p^\mu \sim 0$. These are called soft divergences, since they are due to the interchange of soft (with very low momentum) gluons between external gluons. The second class comes from the region $p^\mu \sim \alpha k_i^\mu$ and are called collinear divergences, since in this case the momentum of the gluon interchanged is proportional to the momentum of one of the external gluons. We can also have soft and collinear divergencies.

² Among the vast literature on the subject we refer the reader to [3][4][5][6] and references therein for a detailed account of the material exposed in these notes.

³ These amplitudes can be used as building blocks for constructing well defined, IR finite, physical observables.

Once a regulator is introduced the amplitudes are finite but will depend explicitly on such regulator. IR divergences manifest as poles in ϵ , *e.g.* $A_n^{(L)} \simeq \frac{1}{\epsilon^{2L}} + \dots$

As already mentioned, the color ordered amplitudes $A_n^{(L)}$ depend also on the helicities of the external gluons. In MSYM, scattering amplitudes satisfy super-symmetric Ward identities that imply the vanishing of the amplitudes for particular choices of the gluon helicities. For instance, it can be shown that the amplitude vanishes when all helicities, or all but one, are plus

$$\mathcal{A}(+++...+) = \mathcal{A}(-++...+) = 0 \quad (2.3)$$

The first non trivial example is the one with two minuses and the rest plus, $\mathcal{A}(- - ++...+)$. Such amplitudes are called maximally helicity violating (MHV) amplitudes.

The simplicity of studying MHV amplitudes is given by the fact that they contain a single Lorentz structure, which is captured by the tree level amplitude, to all orders in perturbation theory. It is then convenient to factor out the tree level amplitude and study the reduced amplitude

$$M_n^{(L)}(\epsilon) = \frac{A_n^{(L)}(\epsilon)}{A_n^{(0)}} \quad (2.4)$$

which depends only on the kinematical invariants and the regulator.

Two loops computations show that at this order, the amplitudes can be written in terms of lower order amplitudes, for instance

$$M_4^{(2)}(\epsilon) = \frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) M_4^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon) \quad (2.5)$$

such relation is non trivial, since the constants $f^{(2)}(\epsilon)$ and $C^{(2)}$ don't depend on the kinematics. Notice that in order to check such relation, $M_4^{(1)}(\epsilon)$ should be computed up to ϵ^2 order. An analogous relation, with the same $f^{(2)}(\epsilon)$ and $C^{(2)}$, is satisfied by the five point scattering amplitude.

Based on explicit computations, plus the well known structure of divergences and consistency checks when taking collinear limits, Bern, Dixon and Smirnov (BDS) [4] proposed the following ansatz for the MHV scattering amplitudes of any number of gluons at any loop order

$$\mathcal{M}_n \equiv 1 + \sum_{\ell=1} \alpha^\ell M_n^{(\ell)} = \exp \left[\sum_{\ell=1} \alpha^\ell \left(f^{(\ell)}(\epsilon) M_4^{(1)}(\ell\epsilon) + C^{(\ell)} + \mathcal{O}(\epsilon) \right) \right] \quad (2.6)$$

where

$$f^{(\ell)}(\epsilon) = f_0^{(\ell)} + \epsilon f_1^{(\ell)} + \epsilon^2 f_2^{(\ell)}, \quad \alpha \approx \lambda \mu^{2\epsilon} \quad (2.7)$$

α is proportional to the 't Hooft coupling constant and keeps track of the perturbation order. The IR regulator scale μ accounts for the fact that in dimensions different from four the coupling constant is not dimensionless.

The structure of the IR divergent terms in (2.6) was determined in [7]. The non-trivial conjecture is that the finite pieces are given by the same functions that characterize the IR divergent terms. Hence, the BDS ansatz give us the (log of the) amplitude at any loop order and for any number of gluons in terms of the one loop amplitude, up to a set of numbers that have to be computed (for instance by computing amplitudes explicitly for a low number of points). We will see that there are good reasons for thinking that BDS is correct for four and five gluons but that starting at six gluons it does not give us the right answer.

We will be particularly interested in the scattering of four gluons, in that case the BDS ansatz reduces to the simple expression

$$\begin{aligned} A_4 &= A_{tree} (A_{div,s})^2 (A_{div,t})^2 \exp \left(\frac{f(\lambda)}{8} \left(\log \frac{s}{t} \right)^2 + const \right) \\ A_{div,s} &= \exp \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) - \frac{1}{4\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) \right) \end{aligned} \quad (2.8)$$

where s and t are the usual Mandelstan variables for the scattering of four particles. The amplitude has two pieces, a divergent one and a finite one. The leading divergent piece is characterized by the so called cusp anomalous dimension $f(\lambda) = (\lambda \partial_\lambda)^2 f^{(-2)}(\lambda)$, while the subleading divergent part is controlled by the function $g(\lambda)$, sometimes called collinear anomalous dimension. Notice that the cusp anomalous dimension also controls the kinematical dependent factor of the finite piece, proportional to $(\log \frac{s}{t})^2$. Much is known about $f(\lambda)$, in particular, by independent means, its strong coupling behavior has been computed [8]

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \dots, \quad \lambda \gg 1 \quad (2.9)$$

In the next section we explain how to compute gluon scattering amplitudes of planar MSYM at strong coupling by using the *AdS/CFT* duality. We will show that the four gluons answer is indeed given by (2.8) at strong coupling.

3. Gluon scattering amplitudes at strong coupling

In order to attack the problem of computing scattering amplitudes at strong coupling we will make use of the *AdS/CFT* duality [9]. This duality, expresses the equivalence between four dimensional MSYM and type IIB string theory on $AdS_5 \times S^5$. The duality provides us with a dictionary between the parameters on both sides

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'}, \quad \frac{1}{N} \sim g_s \quad (3.1)$$

relating in this way the 't Hooft coupling constant to the radius of compactification of the S^5 and AdS_5 in string units, and the inverse of the number of colors to the string coupling constant.

Thus, we see that in the limit of a large number of colors, strings don't split or join, and we can describe string theory by a non linear sigma model. In the regime in which λ is very large, this sigma model is weakly coupled.

As in the gauge theory, we need to introduce a regulator to properly define scattering amplitudes. In order to set-up our computation we introduce a D-brane as IR regulator.

3.1. Set-up of the computation

As a first IR regulator we consider a D-brane localized in the radial direction. In other words, we start with the AdS_5 metric written in Poincare coordinates

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2} \quad (3.2)$$

and we place a D-brane at some fixed large value of $z = z_{IR}$ and extending along the x_{3+1} coordinates. The asymptotic states are open strings that end on the D-brane. We then consider the scattering of these open strings.

The proper momentum of the strings is $k_{pr} = kz_{IR}/R$, where k is the momentum conjugate to x_{3+1} , plays the role of gauge theory momentum and will be kept fixed as we take away the IR cut-off, $z_{IR} \rightarrow \infty$. Therefore, due to the warping of the metric, the proper momentum is very large, so we are considering the scattering of strings at fixed angle with very large momentum. Amplitudes in such regime were studied in flat space by Gross, Mende and Manes, [10,11]. The key feature of their computation is that the amplitudes are dominated by a saddle point of the classical action. In our case we need to consider classical strings on AdS .

We need then to consider a world-sheet with the topology of a disk with vertex operator insertions on its boundary, which correspond to the external states (see fig. 1). Each color ordered amplitude corresponds to a disk amplitude with a fixed ordering of the open string vertex operators.

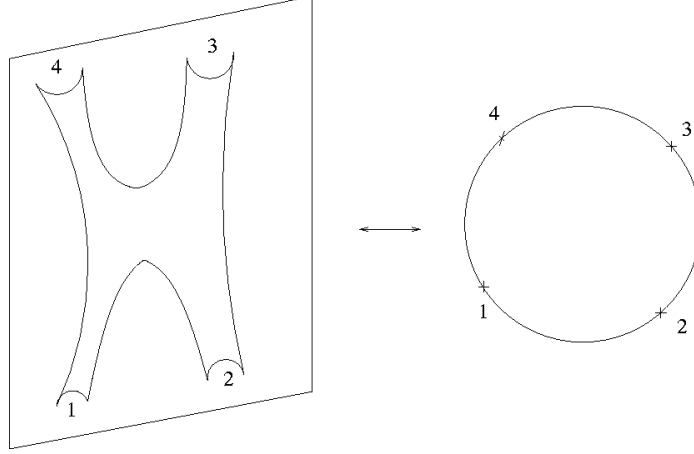


Fig. 1: World-sheet corresponding to the scattering of four open strings.

The boundary conditions for the world-sheet are the following: In the vicinity of a vertex operator, the momentum of the external state fixes the form of the solution and, as the open strings are attached to the D-brane, $z = z_{IR}$ at the boundary.

In order to state more simply the boundary conditions for the world-sheet, it is convenient to describe the solution in terms of T-dual coordinates y^μ , defined as

$$ds^2 = w^2(z)dx_\mu dx^\mu \quad \rightarrow \quad \partial_\alpha y^\mu = iw^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu \quad (3.3)$$

Note that we don't T-dualize the radial direction z . The boundary conditions for the original coordinates x^μ , which are that they carry momentum k^μ , translates into the condition that y^μ has "winding"

$$\Delta y^\mu = 2\pi k^\mu \quad (3.4)$$

After defining $r = R^2/z$ we end up again with the AdS_5 metric

$$ds^2 = R^2 \frac{dy_\mu dy^\mu + dr^2}{r^2} \quad (3.5)$$

Now the boundary of the world-sheet is located at $r = R^2/z_{IR}$ and is a particular line constructed as follows (see fig. 2)

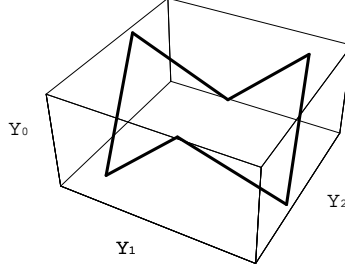


Fig. 2: Polygon of light-like segments corresponding to the momenta of the external particles.

- For each particle of momentum k^μ , draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$.
- Concatenate the segments according to the insertions on the disk (corresponding to a particular color ordering)
- As gluons are massless, the segments will be light-like. Due to momentum conservation, the diagram is closed.

The world-sheet, when expressed in T-dual coordinates, will then end in such sequence of light-like segments (see fig. 3) located at $r = R^2/z_{IR}$

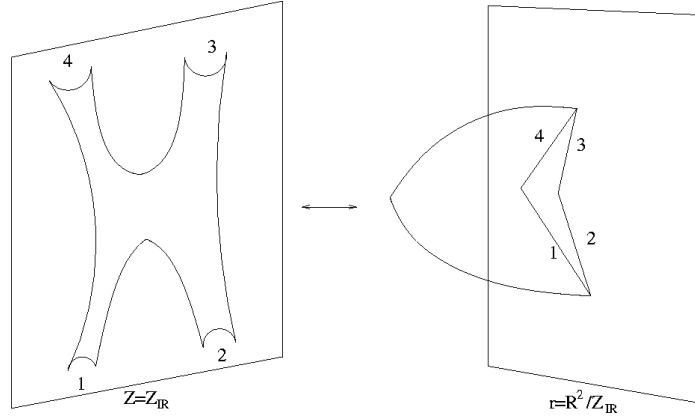


Fig. 3: Comparison of the world sheet in original and T-dual coordinates.

As we take away the IR cut-off, $z_{IR} \rightarrow \infty$, the boundary of the world-sheet moves towards the boundary of the T-dual metric, at $r = 0$. At leading order in the strong coupling expansion, the computation that we are doing is formally the same as the one we would do if we were computing the expectation value of a Wilson loop given by a sequence of light-like segments.

Our prescription is then that the leading exponential behavior of the n -point scattering amplitude is given by the area A of the minimal surface that ends on a sequence of light-like segments on the boundary

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A(k_1, \dots, k_n)} \quad (3.6)$$

The area A contains the kinematical information through its boundary conditions.

We stress that our computation is blind to the polarization of the gluons, which contribute to prefactors in (3.6) and are subleading in $1/\sqrt{\lambda}$.

In the following, we will show in detail how our prescription works for the scattering of four gluons and compare our results with field theory expectations.

3.2. $n = 4$ case

Consider the scattering of two particles into two particles, $k_1 + k_3 \rightarrow k_2 + k_4$ and define the usual Mandelstam variables

$$s = -(k_1 + k_2)^2, \quad t = -(k_2 + k_3)^2 \quad (3.7)$$

According to our prescription we need to find the minimal surface ending in the following light-like polygon

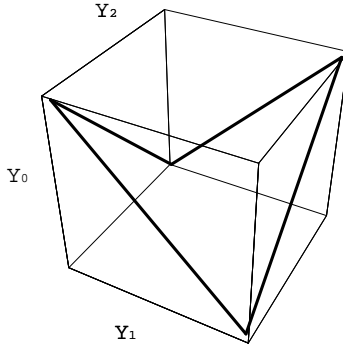


Fig. 4: Polygon corresponding to the scattering of four gluons

In order to write the Nambu-Goto action it is convenient to use Poincare coordinates (r, y_0, y_1, y_2) , setting $y_3 = 0$ and parametrize the surface by its projection to the (y_1, y_2) plane. In this case we obtain an action for two fields, r and t , living in the space parametrized by y_1 and y_2

$$S = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2} \quad (3.8)$$

The classical equations of motion should then be supplemented by the appropriate boundary conditions. We consider first the case with $s = t$ where the projection of the Wilson lines is a square. By scale invariance, we can change the size of the square. We choose the edges of the square to be at $y_1, y_2 = \pm 1$. The boundary conditions are then given by

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1 \quad (3.9)$$

The form of the solution near each of the cusps can be obtained from the single cusp solution of [12]. Making educated guesses satisfying the boundary conditions and with the correct properties near the cusps we propose

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)} \quad (3.10)$$

Remarkably it turns out to be a solution of the equations of motion. However, in order to capture the kinematical dependence of (2.8) we need to consider more general solutions with $s \neq t$. In this case the projection of the surface to the (y_1, y_2) plane will not be an square but a rombus, with s and t given by the square of the distance between opposite vertices, as shown in fig. 5.

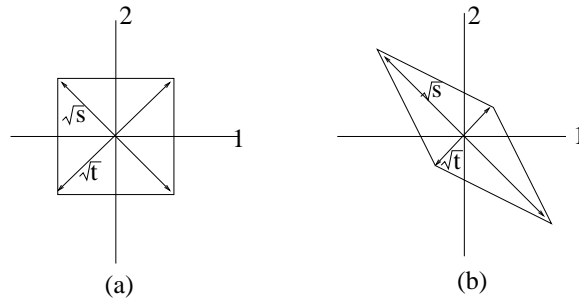


Fig. 5: Projection to the plane (y_1, y_2) of the surface for the cases $s = t$ and $s \neq t$.

In order to find the solution for this more general case, it is instructive to study the surface (3.10) in terms of embedding coordinates. These are coordinates where we view AdS_5 as the following surface embedded in $R^{2,4}$

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4 = -1 \quad (3.11)$$

The relation between these and Poincare coordinates is

$$\begin{aligned} Y^\mu &= \frac{y^\mu}{r}, \quad \mu = 0, \dots, 3 \\ Y_{-1} + Y_4 &= \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r} \end{aligned} \quad (3.12)$$

The surface (3.10) is then given by

$$Y_0 Y_{-1} = Y_1 Y_2, \quad Y_3 = Y_4 = 0 \quad (3.13)$$

Once we have written our solution in embedding coordinates, we notice that we can apply $SO(2,4)$ transformations, that are linearly realized in this coordinates, in order to obtain new solutions. This $SO(2,4)$ symmetry is sometimes referred to as "dual conformal symmetry" and should not be confused with the original $SO(2,4)$ symmetry associated to the original AdS space. It was first observed in computations at weak coupling in [13].

Solutions with $s \neq t$ can be obtained by starting with (3.13) and performing a boost in the 0 – 4 direction

$$Y_0 Y_{-1} = Y_1 Y_2, \quad Y_4 = 0 \quad \rightarrow \quad Y_4 - v Y_0 = 0, \quad \sqrt{1 - v^2} Y_0 Y_{-1} = Y_1 Y_2 \quad (3.14)$$

Hence, we end up with a two parameters solutions, one related to the size of the initial square and another related to the boost parameter.

$$\begin{aligned} r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, & y_0 &= \frac{a \sqrt{1 + b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, & y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \end{aligned} \quad (3.15)$$

where we have written the surface as a solution to the equations of motion in conformal gauge

$$iS = -\frac{R^2}{2\pi} \int \mathcal{L} = -\frac{R^2}{2\pi} \int du_1 du_2 \frac{1}{2} \frac{(\partial r \partial r + \partial y_\mu \partial y^\mu)}{r^2} \quad (3.16)$$

a and b encode the kinematical information of the scattering as follows

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \quad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}, \quad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2} \quad (3.17)$$

According to the prescription, we should now plug the classical solution into the classical action in order to obtain the four point scattering amplitude at strong coupling. However, in doing so, we obtain a divergent answer. That is of course the case, since we have taken the IR regulator away, in order to obtain a finite answer we need to reintroduce a regulator.

i.-Dimensional regularization

Gauge theory amplitudes are regularized by considering the theory in $D = 4 - 2\epsilon$ dimensions. More precisely [14], one starts with $\mathcal{N} = 1$ in ten dimensions and then dimensionally reduce to $4 - 2\epsilon$ dimensions. For integer 2ϵ this is precisely the low energy theory living on a Dp -brane, where $p = 3 - 2\epsilon$. We regularize the amplitudes at strong coupling by considering the gravity dual of these theories. The string frame metric is

$$ds^2 = f^{-1/2} dx_{4-2\epsilon}^2 + f^{1/2} [dr^2 + r^2 d\Omega_{5+2\epsilon}^2], \quad f = (4\pi^2 e^\gamma)^\epsilon \Gamma(2 + \epsilon) \mu^{2\epsilon} \frac{\lambda}{r^{4+2\epsilon}} \quad (3.18)$$

We are then led to the following action

$$S = \frac{\sqrt{c_\epsilon} \lambda \mu^\epsilon}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon} \quad (3.19)$$

Where $\mathcal{L}_{\epsilon=0}$ is the Lagrangian density for AdS_5 . The presence of the factor r^ϵ will have two important effects. On one hand, previously divergent integrals will now converge. On the other hand, the equations of motion will now depend on ϵ and we were not able to compute the full solution for arbitrary ϵ . However, we are interested in computing the amplitude only up to finite terms as we take $\epsilon \rightarrow 0$. In that case, it turns out to be sufficient to plug the original solution into the ϵ -deformed action ⁴. After performing the integrals we obtain:

⁴ Up to a contribution from the regions close to the cusps that add an unimportant additional constant term.

$$S \approx \sqrt{\lambda} \frac{\mu^\epsilon}{a^\epsilon} {}_2F_1 \left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2 \right) \quad (3.20)$$

Expanding in powers of ϵ we get the final answer

$$\begin{aligned} \mathcal{A} = e^{iS} &= \exp \left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi} \left(\log \frac{s}{t} \right)^2 + \tilde{C} \right] \\ S_{div} &= 2S_{div,s} + 2S_{div,t} \\ iS_{div,s} &= -\frac{1}{\epsilon^2} \frac{1}{2\pi} \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon} \frac{1}{4\pi} (1 - \log 2) \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-s)^\epsilon}} \end{aligned} \quad (3.21)$$

This should be compared with the field theory expectations (2.8). We notice that the general structure is in perfect agreement with the BDS ansatz. Once we use the strong coupling behavior for the cusp anomalous dimension (2.9) we see that the leading divergence has the correct coefficient, besides, from (3.21) one could extract the strong coupling behavior of the function $g(\lambda)$. Finally, the kinematical part of the finite piece agrees exactly with the four gluons BDS prediction.

ii.-Radial cut-off

A more common regularization scheme for computing minimal areas in AdS is to introduce a cut-off in the radial direction. The correct procedure is to impose the boundary conditions at some small $r = r_c$. It turns out, however, that in order to compute the finite piece as $r_c \rightarrow 0$ it suffices to use the original solution and cut the integral giving the area at $r = r_c$ ⁵

In order to compute the regularized area for the scattering of four gluons it is convenient to work in conformal gauge. In this case, the problem boils down to compute the area enclosed by the curve

$$\frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} = r_c \quad (3.22)$$

The resulting integrals are pretty tedious. One way to proceed is by expanding the integrand in power series of r_c/a and integrating term by term. Equivalently, one can use Green's theorem and express the area as a one dimensional integral over the boundary of

⁵ The situation is completely analogous that what happened when using dimensional regularization. ■

the world-sheet. Finally, applying various known identities between *ArcSech* and logarithms we arrive to the final expression for the area

$$iS = -\frac{\sqrt{\lambda}}{2\pi}A, \quad A = \frac{1}{4} \left(\log \left(\frac{r_c^2}{-8\pi^2 s} \right) \right)^2 + \frac{1}{4} \left(\log \left(\frac{r_c^2}{-8\pi^2 t} \right) \right)^2 - \frac{1}{4} \log^2 \left(\frac{s}{t} \right) + \text{const.} \quad (3.23)$$

Several comments are in order. First, notice that the structure of infrared divergences is of the form $\log^2(r_c^2/s)$, and the coefficient in front of double logs and the finite piece are the same, and can be identified with the cusp anomalous dimension, as in the case of dimensional regularization. Second, single logs are absent. Finally, the finite piece agrees, up to an additive constant, with the one obtained by using dimensional regularization. Hence, the computation of amplitudes at strong coupling does not need to be done by using dimensional regularization.

The IR structure of amplitudes at strong coupling in the general case of n -point amplitudes can easily be understood. Given the cusp formed by a pair of neighboring gluons with momenta k_i and k_{i+1} we associate the kinematical invariant $s_i = (k_i + k_{i+1})^2$. We expect the following structure for the IR divergent part of the action

$$iS_{div} = -\frac{\sqrt{\lambda}}{2\pi} \sum_i I\left(\frac{r_c^2}{s_{i,i+1}}\right) \quad (3.24)$$

where $I(\frac{r_c^2}{s_{i,i+1}})$ can be computed following [15] but using a radial cut-off instead of dimensional regularization.

$$4I = \int_{\xi}^1 \int_{\frac{\xi}{X^-}}^1 \frac{1}{X^- X^+} = \frac{1}{2} \log^2 \xi, \quad \xi = \frac{r_c^2}{-8\pi^2 s_{i,i+1}} \quad (3.25)$$

Hence, when using a radial cut-off as regulator, we expect the following structure for scattering amplitudes at strong coupling

$$iS_n = -\frac{\sqrt{\lambda}}{16\pi} \sum_{i=1}^n \log^2 \left(\frac{r_c^2}{-8\pi^2 s_{i,i+1}} \right) + \text{Fin}(k_i) \quad (3.26)$$

It is easy to check that the general form of the amplitude for the case $n = 4$ is consistent with this general expression.

For the discussion below, it will be important to consider a radial cut-off that depends on the point at the boundary we are approaching, *i.e.* $r_c(x)$. In that case, the structure of the amplitude turns out to be as follows

$$iS_n = -\frac{\sqrt{\lambda}}{16\pi} \sum_{i=1}^n \log^2 \left(\frac{r_c^2(x_i)}{-8\pi^2 s_{i,i+1}} \right) + \text{Fin}(k_i) + \sum_{i=1}^n E_{edge}^i(r_c) \quad (3.27)$$

The last sum in this expression corresponds to finite extra contributions coming from the edges

$$E_{edge}^i = \frac{\sqrt{\lambda}}{2\pi} \int_0^1 \frac{ds}{s} \log \left(\frac{r_c(s)r_c(1-s)}{r_c(0)r_c(1)} \right) \quad (3.28)$$

where s running from zero to one parametrizes the i th edge, namely $x^\mu(s) = x_i^\mu + s(x_{i+1}^\mu - x_i^\mu)$ and $r_c(s)$ is a shorthand notation for $r_c(x(s))$. For instance, a simple example is that of a cut-off that takes the value $r_c(x_i)$ at the i th cusp and varies linearly between cusp and cusp, in this case

$$E_{edge}^i = \frac{\sqrt{\lambda}}{4\pi} \log^2 \frac{r_c(x_i)}{r_c(x_{i+1})} \quad (3.29)$$

3.3. Conformal Ward identity ⁶

An important ingredient of the computation for the case $n = 4$ was the existence of a dual $SO(2,4)$ symmetry. This symmetry allowed the construction of new solutions and fixed somehow the finite piece of the scattering amplitude. Naively, this conformal symmetry would imply that the amplitude is independent of s and t , since they can be sent to arbitrary values by a dual conformal symmetry. The whole dependence on s and t arises due to the necessity of introducing an IR regulator. However, we will see that, after keeping track of the dependence on the IR regulator, the amplitude is still determined by the dual conformal symmetry.

To a symmetry we associate a Ward identity, that will impose certain constraints on scattering amplitudes. In order to understand these constraints for the case of the dual $SO(2,4)$ symmetry it is convenient to consider the amplitude regulated by a radial cut-off.

Given the momenta k_i of the external gluons, the boundary of the world-sheet contains cusps located at x_i , with $2\pi k_i = x_i - x_{i+1}$. Now imagine that we regularize the area by choosing a cut-off r_c . Moreover, we would like this cut-off to depend on the point at the boundary we are approaching, *i.e.* $r_c \rightarrow r_c(x)$. From the discussion above we expect the regulated area to have the general form

⁶ The original idea leading to the argument below is due to Juan Maldacena.

$$A_n^{reg} = f \sum_{i=1}^n \log^2 \left(\frac{r_c^2(x_i)}{-2x_{i-1,i+1}^2} \right) + Fin(x_i) \quad (3.30)$$

where we have disregarded the extra terms coming from the edges since they can be seen not to affect the following argument ⁷. $SO(2,4)$ transformations will then act on the points x_i and $r_c(x_i)$. By requiring the area to be invariant under the action of special conformal transformations

$$K^\mu A_n^{reg} = \left(\sum_{i=1}^n 2x_i^\mu (x_i \cdot \partial_{x_i} + r(x_i) \partial_{r(x_i)}) - x_i^2 \partial_{x_i^\mu} \right) A_n^{reg} = 0 \quad (3.31)$$

we get an equation for the finite piece of the amplitude. This same equation was obtained perturbatively to all lops in relation with expectation values of Wilson loops in [16] and at strong coupling by using dimensional regularization in [17].

Supposing that the dual conformal symmetry is present beyond the strong coupling limit, a similar argument can be extended to all values of the coupling, *e.g.* by using as a regulator an energy scale $\mu(x)$ and assuming that the amplitude has divergences which depend only on $\mu(x)$ at the cusps (or assuming that special conformal transformations annihilate the extra pieces coming from the edges, as it happened at strong coupling.)

It turns out, that for the case of $n = 4$ and $n = 5$, this equation fixes uniquely the form of the finite piece, to be the one in the BDS conjecture. At this point we do not know if the dual conformal symmetry is an exact property of planar amplitudes. We do know, however, that it is a symmetry of all the weak and strong coupling computations that have been done so far. If we assumed that it is a symmetry, then we conclude that the BDS conjecture for four and five gluons is correct.

4. Scattering amplitudes vs. Wilson loops

The computation of the previous section shows a relation between two seemingly different quantities, scattering amplitudes and expectation values of Wilson loop.

⁷ One can convince oneself, for instance, by considering the simplified case (3.29) and applying the generator of special conformal transformations to such extra terms. It is also instructive to apply the generator of dual conformal transformations, whose relevant piece is of the form $\int ds x^\mu(s) r_c(s) \frac{\delta}{\delta r_c(s)}$, to the general extra terms (3.28) and compare this expression to eq. (34) of [16].

More precisely, one can consider the planar, MHV amplitude for n gluons, $A(k_1, \dots, k_n)$ and an associated Wilson loops in position space, $W(x_1, \dots, x_n)$ formed by light-like segments joining cusps at x_i , with $2\pi k_i = x_i - x_{i+1}$. The results of the previous section imply that both quantities are equivalent at strong coupling. Quite remarkable, explicit computations [18][19] show that this duality continues to hold also at weak coupling! The duality between amplitudes and Wilson loops would imply the dual conformal symmetry, since the dual conformal invariance becomes the ordinary conformal invariance of the Wilson loop computation.

Beyond explicit computations at one loop for any number of gluons and two loops computations to be mentioned later, expectation values of Wilson loops were shown to possess the dual conformal symmetry and to satisfy the same conformal ward identity (3.31) [20][16]. Such dual conformal symmetry was also observed in perturbative scattering amplitudes [4][21][22] (though it is not a proven symmetry).

The equivalence of both quantities at one loop can be stated as follows. The BDS ansatz (which to one loop is correct by construction) can be written as

$$\log M_n = Div_n + \frac{f(\lambda)}{4} a_1(k_1, \dots, k_n) + h(\lambda) + nk(\lambda) \quad (4.1)$$

with a_1 the one loop amplitude and $h(\lambda)$ and $k(\lambda)$ functions that are independent on the kinematics and the number of gluons and we are not interested on them. On the other hand, we can compute the one-loop expectation value of the associated Wilson loop

$$\langle W_n \rangle = \tilde{Div}_n + w_1(k_1, \dots, k_n) + c(\lambda) + nd(\lambda) \quad (4.2)$$

where $c(\lambda)$ and $d(\lambda)$ are functions that are independent on the kinematics or the number of edges and are not interesting for us. Then, explicit computations show that $a_1 = w_1$.

Summarizing what we have said up to now:

- The BDS ansatz implies that the strong coupling limit of MHV planar scattering amplitudes is given by the one loop amplitude times the strong coupling limit of the cusp anomalous, that is $a_{strong}(k_1, \dots, k_n) = f^{strong} a_1(k_1, \dots, k_n)$.
- Our computation implies that scattering amplitudes and expectation values of Wilson loops agree at strong coupling, namely $a_{strong}(k_1, \dots, k_n) = w_{strong}(k_1, \dots, k_n)$.
- While explicit one loop computations show that $a_1(k_1, \dots, k_n) = w_1(k_1, \dots, k_n)$.

Assuming BDS and using the next two results we arrive to

$$w_{strong}(k_1, \dots, k_n) = f^{strong} w_1(k_1, \dots, k_n) \quad (4.3)$$

So, the expectation value of a Wilson loop at strong coupling is given by its one loop expectation value times the cusp anomalous dimension at strong coupling. Obviously, that is a very non trivial statement, and indeed, for the case of four and five edges, that is the case! However, as we have seen, in this case the expectation value is fixed by symmetries. Hence, in order to test the BDS conjecture, we need to consider polygons with more than five edges.

4.1. Testing BDS

As just mentioned, in order to test the BDS conjecture, one would need to consider polygons of more than five edges. It seems very difficult to find explicit solutions for six edges. However, we consider a zig-zag configuration with a large number of edges that approximates the rectangular Wilson loop.

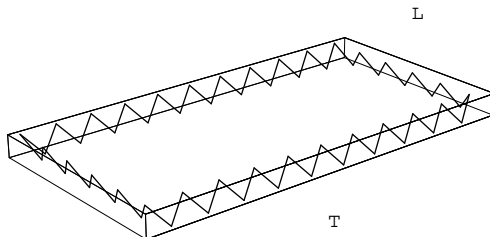


Fig. 6: Zig-Zag configuration approaching the space-like rectangular Wilson loop.

In the limit of very large T and L and for $T \gg L$, we can compute the expectation value both at weak and strong coupling [23][24], obtaining

$$\log \langle W_{rect}^{weak} \rangle = \frac{\lambda}{8\pi} \frac{T}{L}, \quad \log \langle W_{rect}^{strong} \rangle = \frac{\sqrt{\lambda} 4\pi^2}{\Gamma(1/4)^4} \frac{T}{L} \quad (4.4)$$

While the BDS ansatz prediction would be $\log \langle W_{rect}^{strong} \rangle = \frac{\sqrt{\lambda}}{4} \frac{T}{L}$. Hence, the BDS ansatz needs to be revised for a large number of gluons.

The previous reasoning can be also repeated by considering the two loops amplitude versus the two loops Wilson loop expectation values. Explicit two loops computations for the rectangular Wilson loop, show that at this order and for a large number of gluons, either BDS or the duality between scattering amplitudes and Wilson loops fails.

4.2. $n = 6$ case

As explained above, the presence of the dual conformal symmetry fixes both the scattering amplitudes and the expectation values of the Wilson loops for the case of $n = 4$ and $n = 5$, and in both cases, the result agrees with the BDS ansatz.

The BDS ansatz for six gluons satisfies the dual conformal Ward identities, however, it is not uniquely fixed by these. The general solution is the BDS ansatz plus an arbitrary function of invariant cross ratios

$$\begin{aligned} K.f(u_1, u_2, u_3) = 0 &\rightarrow A_6 = A_{BDS} + f(u_1, u_2, u_3), \\ u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} \end{aligned} \tag{4.5}$$

Note that this invariant cross ratios cannot be constructed for $n < 6$. A remarkable explicit computation for the scattering of six gluons at two loops [25], shows that indeed $f \neq 0$ and hence the BDS ansatz is to be modified for six gluons at two loops.

A parallel computation for the two loops expectation value of the associated Wilson loops has also been carried out [26]. Quite remarkably, the duality between scattering amplitudes and Wilson loops continues to hold for this case! [27][25]. This is a strong indication that the duality may be true for any number of gluons at any loop order.

5. Conclusions

In this notes we have described recent progress in computing planar scattering amplitudes on $\mathcal{N} = 4$ SYM at strong coupling by using the AdS/CFT correspondence. The computation reduces to a minimal surface problem in AdS , with boundary conditions fixed by the momenta of the external particles.

Amplitudes are IR divergent and a regulator needs to be introduced in order to define them properly. We perform explicit computations both by using dimensional regularization and a cut-off in the radial direction. While the former scheme allows a direct comparison with gauge theory results, the former is more convenient for understanding the conformal properties of the amplitudes.

One of our main motivations was to test the BDS ansatz. Our results agree with this conjecture for $n = 4, 5$ but disagree for a large number of gluons. The agreement can be understood as due to the dual conformal symmetry. Moreover, explicit computations indeed show that the BDS ansatz is not correct for six gluons at two loops.

An important ingredient in the computation of amplitudes at strong coupling is the presence of a dual $SO(2, 4)$ conformal symmetry. We presented a simple argument leading to a Ward identity for this symmetry. In addition, The strong coupling picture suggests a relation between amplitudes and Wilson loops, which seems to be a true relation and it survived the nontrivial check of [27][25].

There are many directions one could try to follow

- Construct new solutions corresponding to the scattering of more than four gluons. Despite some partial progress [28][29][30][31], general solutions other than the one for $n = 4$ are missing. Such solutions would be very useful in trying to understand the existence of iterative relations, from the strong coupling side.
- Try to use the machinery of integrability in order to find new solutions, or the value of the action even without knowing the classical solutions. Besides, integrability may provide a set of constraints that would fix the form of the amplitudes.
- Compute subleading corrections in $1/\sqrt{\lambda}$. Among other things, one should be able to compute the dependence on the helicities of the particles and understand the particular role played by MHV amplitudes. Some attempts have been done in [32], where apparently is subtle to extend dimensional regularization beyond the classical analysis. Maybe it is convenient to consider other schemes, like a radial cut-off for instance.
- Try to understand higher genus corrections. These would correspond to non planar corrections to scattering amplitudes.
- The extension of the prescription described here to other backgrounds, less supersymmetric or without conformal invariance, is also an important problem. See [33][34][35][36] for recent interesting developments in this direction.
-
- An interesting problem would be to try to determine the duality between MHV scattering amplitudes and expectation values of Wilson loops. In case this duality holds true, it would be interesting to extend this equivalence to non MHV amplitudes.
- The apparent equivalence between scattering amplitudes and Wilson loops for the case of six legs, hints to the existence of new symmetries that would fix the form of both quantities in this case.
- Finally, it would be very interesting to find appropriate modifications to the BDS ansatz, in order to describe higher point amplitudes at all values of the coupling. Though it would be very surprising to find a general explicit formula.

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